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ON APPROXIMATE SOLUTION OF NONLINEAR WEAKLY SINGULAR VOLTERRA INTEGRAL EQUATION

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ABSTRACT

The paper is concerned with the numerical analysis of anonlinear weakly singular Volterra integral equation by the product Euler's method, we proved that the error in Euler's method is of order $O(h)$. To illustrate the convergence rates some numerical results are included confirming the theoretical estimates.

Keywords: an onlinear weakly singular Volterra integral equation; Abel Type integral equation; the quadrature error; the product Euler's method**.**

Mathematics Subject Classification(2010): 45G05, 45E10,45Dxx

INTRODUCTION

Abel's equation is one of the integral equations derived directly from a concrete problem of physics, without passing through a differential equation. This equation was applied by Niels Abel in 1823 to describe a sliding point mass in a vertical plane on a unknown curve under gravitational force. The point mass starts its motion without initial velocity from a point which has a vertical distance *x* from the lowest point of the curve see Wazwaz (1997). Using the work-energy theorem, the equation of the unknown curve that obtained is the well-known Abel integral equation

$$f(x) = \int_0^x \frac{u(t)}{\sqrt{2g(x-t)}} dt \qquad (1)$$

Where $f(x)$ is a given function, $u(t)$ is an unknown function in which g is the acceleration due to gravity. This equation is a particular case of a linear Volterra equation of the first kind. After one century, in 1924, Zeilon (1924) studied the generalized Abel's integral equation on a finite segment. Many works have already been done to solve the Abel integral equations. The brief history and basic classical solution to Abel problems are well covered in the monographs by Muskhelishvilli (1963), Polyanin and Zaitsev (1999). Well-known examples of statistical inverse problems are Wicksells problem in stereology, where one wishes to recover the frequency distribution of actual radius of spherical particles from a sample of planar cuts see (Ripley, 1981; Stoyan *et al*. 1987) and computerized tomography, where the density of a body is to be recovered from a collection of line integrals see (Vardi *et al*. 1985; Hall *et al*. 2003). Other examples range from inverse heat conduction to a wide variety of problems involving de-convolution and visualizing. In statistical terminology these problems might be classified as those of indirect curve estimation. Anderssen (1980), Anderssen and De Hoog(1990), Gorenflo and Vessella (1991), Gorenflo *et al*. (1997) are reviews of methods based on Abel integral equations and Carroll *et al*. (1991) for a review of theoretical aspects of ill-posed problems in statistics.

Huang *et al*. (2008), Capobianco and Conte (2006), Yousefi (2006), Chakrabarti (2008), Gorenflo and Luchko (1997) have also solved Abel integral equations by using Taylor expansion, waveform relaxation method, numerical solution by Legendrewavelet, direct function theoretic method and operational matrix respectively. Another study, Singh *et al*. (2010) have used Bernstein operational method for solving Abel integral equation of second kind. In the last two decades, many powerful and simple methods have been proposed and applied successfully to approximate various types of linear and nonlinear singular integral equations with a wide range of applications. The generalized nonlinear Abel integral equation see Estrada (2000) and Kanwal (1997) is of the form

$$\left[f(t) = \int_0^t \frac{F(u(x))}{(t-x)^{\alpha}} dx, \qquad 0 < \alpha < 1, \right] \qquad (2)$$

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where α is a known constant such that $0 < \alpha < 1$, $f(t)$ is a given function, and $F(u(x))$ is a nonlinear function of $u(x)$. The nonlinear Abel integral equation is a special case of the generalized equation where $\alpha = 1/2$. The expression$(x - t)^{-\alpha}$is called the kernel of the integral equation. The Laplace transform Method see Muhammad *et al*. (2013), Homotopy Perturbation method see Sunil *et al*. (2011), the new iterative method see Praveen (2012) and anther methods are used to handle the generalized nonlinear Abel integral equation (2). The nonlinear weakly-singular Volterra integral equations of the second kind are given by

$$u(t) = f(t) + \int_0^t \frac{F(u(x))}{(t-x)^{\alpha}}dx, \qquad 0 < \alpha < 1,\ x \in [0,T], \qquad (3)$$

This equation arises in many mathematical physics and chemistry applications such as stereology, heat conduction, crystal growth and the radiation of heat from a semi-infinite solid. It is also assumed that the function $f(x)$ is a given real valued function. The nonlinear weakly-singular equation(3) falls under the category of singular equations with singular kernel. In Teresa *et al*. (2005) is concerned with the numerical solution of nonlinear weakly singular Volterra integral equation with type kernel of the form $p(t,s,y(s))(t-s)^{-\alpha}$, with $\alpha = 2/3$ and $p(t,s,y(s)) = s^{1/3}y^4$, with a non-smooth solution by investigating the application of several numerical methods.

In the present work we investigate the application of the product Euler's method to an onlinear weakly-singular Volterra integral equation of second kind which describes the temperature in a semi-infinite solid, whose surface can dissipate heat by nonlinear radiation. At the surface, energy is supplied according to the given function $f(x)$, while radiated energy see Olmstead and Handelsman (1976), escapes in proportion to $u^n(x)$.

$$u(t) = \frac{1}{\sqrt{\pi}}\int_0^t \frac{f(x) - u^n(x)}{\sqrt{t-x}}dx \qquad (4)$$

Where $u(t)$ gives the temperature at the surface for all time. In principle, equation (4) can be solved by Adomian decomposition method handles such problems effectively by selecting the values of $f(x)$ and n see Wazwaz (2011). For selecting $f(x) = 1/2$ and $n=4$ equation (4) becomes

$$u(t) = \frac{\sqrt{t}}{\pi} - \frac{1}{\sqrt{\pi}}\int_0^t \frac{u^4(x)}{\sqrt{t-x}}dx, \qquad t \in [0,1] \qquad (5)$$

Using the Adomain assumptions for $u(x)$ and $u^4(t)$ the series representation for the solution of equation (5) is given by

$$u(t) = t^{1/2}\begin{pmatrix} 0.564 - 0.061t^2 + 0.020t^4 - 0.008t^6 \\ + 0.004t^8 - 0.002t^{10} + ... \end{pmatrix},$$

that governs the radiation of heat from a semi-infinite solid having a constant heat source.

For selecting $f(x) = 2\sqrt{\frac{x}{\pi}}$ and n=3, equation (4) becomes

$$u(t) = t - \frac{1}{\sqrt{\pi}}\int_0^t \frac{u^3(x)}{\sqrt{t-x}}dx, \qquad t \in [0,1] \qquad (6)$$

Using the Adomain assumptions for $u(x)$ and $u^3(t)$ the series representation for the solution of equation (6) is given by

$$u(t) = t - 0.5158t^{7/2} + 0.6188t^6 - 0.8972t^{17/2} + 1.4142t^{11} - 2.3357t^{27/2} + ...,$$

It is straightforward to demonstrate that (5) and (6) have a unique continuous solution $u(t)$ for $t \in [0,1]$. In the Numerical method section we apply the product Euler's method to equation (5) and prove that it is convergent of order $O(h^{1/2})$. By a detailed analysis we are able to show that, away from origin, the error in Euler's method is of order $O(h)$. Numerical results are presented illustrating the performance of the method in Numerical method section.

MATERIALS AND METHODS

Numerical method

In order to approximate the solution $u(t)$of equation (5), let us introduce the uniform grid X_h On $[0,1]$, with step size $h = 1/N$,

$$X_h = \{t_i = ih, \quad 0 \le i \le N\}.$$

The Euler's method is defined as follows

$$\begin{cases} u_0(t) = \sqrt{\dfrac{t}{\pi}} \\ u_i = \sqrt{\dfrac{t}{\pi}} - \dfrac{1}{\sqrt{\pi}} \displaystyle\sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \frac{dx}{(t_j - x)^{1/2}} u_j^4, \quad i = 1,2,\ldots, N \\ \quad = \sqrt{\dfrac{t}{\pi}} - \dfrac{1}{\sqrt{\pi}} \displaystyle\sum_{j=0}^{i-1} W_{ij} u_j^4 \end{cases} \qquad (7)$$

Where u_i denotes an approximation to $u(t_i)$ and the weights W_{ij} are such that

$$W_{ij} = \int_{t_j}^{t_{j+1}} \frac{dx}{(t_j - x)^{1/2}} \le M_1 \frac{h^{1/2}}{(i-j)^{1/2}}. \qquad (8)$$

The total error $e_i = u(t_i) - u_i$ of the approximate solution of (5) at $t = t_i$ satisfies

$$|e_i| = |u(t_i) - u_i| \le |T_i| + M_2 h^{1/2} \sum_{j=0}^{i-1} \frac{1}{(i-j)^{1/2}} |e_j|, \quad i = 1,\ldots, N \qquad (9)$$

Where T_i is the quadrature error at $t = t_i$, given by:

$$T_i = \frac{1}{\sqrt{\pi}} \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \left(u^4(x) - u^4(t_j)\right) \frac{dx}{(t_j - x)^{1/2}}. \qquad (10)$$

Lemma 2.1. The solution u to equation (5) satisfies the inequality

$$|u(z) - u(z')| \le M_3 |z - z'|^{1/2}, \qquad \forall z, z' \in [0,1] \qquad (11)$$

Where $|z - z'|$ is sufficiently small and B_1 is a positive constant that does not depend on z or z'. **Proof.**

$$\begin{aligned} |u(z) - u(z')| &= \frac{1}{\sqrt{\pi}} \left| \int_0^{z'} \frac{u^4(x)}{(z-x)^{1/2}} dx - \int_0^{z'} \frac{u^4(x)}{(z'-x)^{1/2}} dx + \int_{z'}^{z} \frac{u^4(x)}{(z-x)^{1/2}} dx \right| \\ &\le \frac{1}{\sqrt{\pi}} \left| \int_0^{z'} \frac{u^4(x)}{(z-x)^{1/2}} dx - \int_0^{z'} \frac{u^4(x)}{(z'-x)^{1/2}} dx \right| + \frac{1}{\sqrt{\pi}} \left| \int_{z'}^{z} \frac{u^4(x)}{(z-x)^{1/2}} dx \right| \\ &\le \frac{4M^4}{\sqrt{\pi}} |z - z'|^{1/2} \end{aligned}$$

Where $M = \max_{x \in [0,1]} |u(x)|$.

Lemma 2.2. the quadrature error, T_i satisfies

$$|T_i| \le M_4 h^{1/2}, \qquad i = 1,2,\ldots, N, \qquad (12)$$

Proof. Using the Lipschitz continuity of the function u^4 and lemma 2.1, we get

$$\begin{aligned} |T_i| &\le \left| \frac{1}{\sqrt{\pi}} \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \left(u^4(x) - u^4(t_j)\right) \frac{dx}{(t_j - x)^{1/2}} \right| \\ &\le \frac{1}{\sqrt{\pi}} M_3 L h^{1/2} \cdot \left| \sum_{j=0}^{i-1} \int_{t_j}^{t_{j+1}} \frac{dx}{(t_j - x)^{1/2}} \right| \le M_4 h^{1/2} \end{aligned}$$

Where $|u^4(x) - u^4(t_j)| \le L |u(x) - u(t_j)|$

Using Lemma 2.2 in the inequality (9), and applying a standard weakly singular discrete Gronwall inequality see Dixon and Mckee (1986), leads to the following theorem.

Theorem 2.1. Let $u(t)$ be the solution of (5) and u_i an approximation to $u(t)$ at

t = t_i . Then the error $e_i = u(t_i) - u_i$ satisfies*:*

$\|e\|_\infty \le M_5 h^{1/2}$ (13)

where M_5 is a constant independent of h.

From (13) we have that Euler's method for equation (5) converges with order *1/2<1*. Now, we prove that, at points t_i away from the origin, first order of convergence is achieved. This requires a detailed analysis of the quadrature error, as it was done in Teresa *et al*. (2005) for the product Euler's method, we have the following result.

Lemma 2.3. Teresa *et al*. (2005)The quadrature error, T_i, satisfies

$$|T_i| \le M_6\left(\frac{h^{3/2}}{t_i^{1/2}} + h\right), \qquad i = 1,2,..., N \quad (14)$$

where M_6 is a constant independent of h.

Aconclusion on the convergence order of Euler's method will require the following discrete lemma from Dixon (1985).

Lemma 2.4. Teresa *et al*. (2005) Let $x_i, 0 \le i \le N$, be a sequence of non-negative real numbers satisfying

$$x_i \le \lambda + \frac{\theta}{(ih)^\alpha} + \zeta h^{1-\alpha} \sum_{j=o}^{i-1} \frac{x_i}{(i-j)^\alpha}, \qquad 0 \le i \le N \quad (15)$$

Where $0 \le \alpha \le 1, \lambda, \theta$ are non-negative constants and ζ is a positive constant independent of $h\ (h > 0)$. Then for any $T > 0$ there exists $C = C(T)$ such that

$$x_i \le C\left(\lambda + \frac{\theta}{(ih)^\alpha}\right) \quad 0 \le i \le N \quad (16)$$

Whenever $Nh \le T$.

By applying Lemma 2.4, with $\alpha = 1/2,\ \theta = h^{3/2}\ and\ \zeta = h$, leads to the following theorem.

Theorem 2.2. Let $u(t)$ be the solution of (5) and u_i an approximation to $u(t)$ at $t = t_i$ defined by (7). Then the error $e_i = u(t_i) - u_i$ satisfies

$$|e_i| \le CM_6\left(\frac{h^{3/2}}{t_i^{1/2}} + h\right), \qquad i = 1,2,..., N \qquad (17)$$

where C is a positive constant independent of h. Therefore we can conclude that, the order of the error of Euler's method at the fixed point t_i , away from the origin, is one.

RESULTS AND DISCUSSION

Numerical results. In this section we present some numerical results obtained with the product Euler method considered in the previous section. Tables 1,2 contain approximations of $u(t)$ for equations (5) and (6) respectively obtained with several values of the step size. In Tables 3,4, we have computed experimental rates of convergence for equations (5) and (6) respectively, defined by

$$p \approx \frac{\log\left(\frac{u^{h/2} - u^h}{u^{h/4} - u^{h/2}}\right)}{\log 2} \qquad (18)$$

where $u^h, u^{h/2}$ and $u^{h/4}$ denote approximations to $u(t)$ using the mesh spacing $h, h/2$ and $h/4$, respectively. The results of Tables 3,4 indicate first order of convergence, which is in agreement with the result (17) of Theorem 2.2. In order to obtain error estimates we have taken $u^{1/640}$ as the exact solution of equations (5) and (6). The computed error norms, given by:

$$\|e_N\|_\infty = \max_{1 \le i \le N} \left|u(t_i) - u_i^{1/N}\right|,\ t_i = \frac{i}{N}$$

are displayed in Tables 5,6 together with the corresponding rates of convergence. The numerical results suggest that the global order of convergence is $1/2$, confirming the theoretical result (13).

Table 1. Approximations of u(t) for equation (5).

t	N=80	N=160	N=320	N=640
0.0	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00
0.1	1.783898E-01	1.784011E-01	1.784068E-01	1.784096E-01
0.2	2.522493E-01	2.522813E-01	2.522973E-01	2.523053E-01
0.3	3.089019E-01	3.089606E-01	3.089900E-01	3.090047E-01
0.4	3.566441E-01	3.567344E-01	3.567796E-01	3.568022E-01
0.5	3.986896E-01	3.988160E-01	3.988791E-01	3.989107E-01
0.6	4.366873E-01	4.368533E-01	4.369363E-01	4.369779E-01
0.7	4.716164E-01	4.718256E-01	4.719302E-01	4.719825E-01
0.8	5.043493E-01	5.043813E-01	5.043973E-01	5.044053E-01
0.9	5.348440E-01	5.349344E-01	5.349796E-01	5.350022E-01
1.0	5.636164E-01	5.638256E-01	5.639302E-01	5.639825E-01

Table 2. Approximations of u(t) for equation (6).

t	N=80	N=160	N=320	N=640
0.0	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00
0.1	1.783901E-01	1.784013E-01	1.784068E-01	1.784096E-01
0.2	2.521871E-01	2.522502E-01	2.522817E-01	2.522975E-01
0.3	3.086717E-01	3.088455E-01	3.089325E-01	3.089759E-01
0.4	3.561112E-01	3.564680E-01	3.566464E-01	3.567356E-01
0.5	3.976956E-01	3.983189E-01	3.986306E-01	3.987864E-01
0.6	4.350528E-01	4.360361E-01	4.365277E-01	4.367735E-01
0.7	4.691437E-01	4.705893E-01	4.713120E-01	4.716735E-01
0.8	5.016954E-01	5.023189E-01	5.026306E-01	5.027864E-01
0.9	5.311430E-01	5.325890E-01	5.333120E-01	5.336735E-01
1.0	5.590528E-01	5.600361E-01	5.605277E-01	5.607735E-01

Table 3. Convergence rate for several values of N for equation (5).

t	N=80,160,320	N=160,320,640
0.1	0.98729	1.02554
0.2	1.00000	1.00000
0.3	0.99754	1.00000
0.4	0.99840	1.00000
0.5	1.00912	0.99772
0.6	1.00000	0.99653
0.7	1.00000	1.00000
0.8	1.00000	1.00000
0.9	1.00000	1.00000
1.0	1.00000	1.00000

Table 4. Convergence rate for several values of N for equation (6).

t	N=80,160,320	N=160,320,640
0.1	1.02599	0.97401
0.2	1.00288	0.995427
0.3	0.998340	1.03355
0.4	1.00000	1.00000
0.5	0.99977	1.00046
0.6	1.00015	1.00000
0.7	0.99990	0.99940
0.8	1.00023	1.00046
0.9	1.00000	1.00000
1.0	1.00014	1.00000

Table 5. Errors and convergence rates for equation (5).

N	h	$\|e\|_\infty$	rate
40	0.025	0.3123	0.3621
80	0.0125	0.0782	0.4412
160	0.00625	0.0561	0.5423
320	0.003125	0.0342	0.5331
640	0.0015625	0.0231	0.5201

Table 6. Errors and convergence rates for equation (6).

N	H	$\|e\|_\infty$	rate
40	0.025	0.3043	0.3562
80	0.0125	0.0684	0.3973
160	0.00625	0.0426	0.4923
320	0.003125	0.0265	0.5123
640	0.0015625	0.0165	0.5576

CONCLUSION

We have introduced a nonlinear Volterra integral equations with an Abel type kernels of the $u^4(t-x)^{-1/2}$, $u^3(t-x)^{-1/2}$ respectively. We have shown that, while near the originthe exact solution u(t) behaves like $t^{1/2}$, it is differentiable if t is large enough. Numerical approximations to u were obtained by Euler's method and shown to beconvergent of order *1/2*. Moreover it was proved that, for t away from the origin, the convergence order is one. These results were confirmed by some numerical examples.

REFERENCES

Anderssen, RS. and De Hoog, FR. 1990. Abel integral equations. In: Numerical Solution of Integral Equations, Mathematical Concepts and Methods in Science and Engineering. Ed. Goldberg, MA. Plenum, New York, USA. 42:373-410.

Anderssen, RS. 1980. On the use of linear functionals for Abel-type integral equations in applications. In: Application and Numerical Solution of Integral Equations. Proceedings of a Seminar held at the Australian National University, Canberra. Monographs and Textbooks on Mechanics of Solids and Fluids: Eds. Anderssen, RS., De Hoog, FR. and Lukas, MA. Mechanics and Analysis, Nijhoff, The Hague.6: 195-221.

Chakrabarti, A. 2008. Solution of the Generalized Abel Integral Equation. J. Integral Equations Appl. 20:1-11.

Capobianco, G. and Conte, D. 2006. An efficient and fast parallel method for Volterra integral equations of Abel type. J. Comput. Appl. Math. 189:481-493.

Carroll, RJ., Van Rooij, ACM. and Ruymgaart, FH. 1991. Theoretical aspects of ill-posed problems in statistics. Acta Appl. Math. 24:113-140.

Dixon, J. 1985. On the order of the error in discretization methods for weakly singular second kind Volterra integral equations with non-smooth solutions. BIT. 25:624-634.

Dixon, J. and McKee, S. 1986. Weakly singular discrete inequalities. ZAMM. 66: 535-544.

Estrada, R. and Kanwal, R. 2000. Singular Integral Equations, Birkhauser, Berlin. 43-69.

Gorenflo, R., Medvedeva, T. and Rubin, B. 1997. Locally controllable regularization of Abel integral equations via operator complementation. J. Inv. Ill-Posed Pr. 5:427-436.

Gorenflo, R. and Luchko, Y.1997. Operationl method for solving generalized Abel integral equation of second kind. Integral Transforms and Special Functions. 5:47-58.

Gorenflo, R. and Vessella, S. 1991. Abel Integral Equations. Analysis and Applications. In: Lecture Notes in Mathematics. Springer, New York, USA. 1461:35-63.

Hall, P., Paige, R. and Ruymgaart, FH. 2003. Using wavelet methods to solve noisy Abel-type equations with discontinuous inputs. Journal of Multivariate Analysis. 86: 72-96.

Huang, L., Huang, Y. and Li, XF. 2008. Approximate solution of Abel integral equation. Comput. Math. Appl. 56:1748-1757.

Kanwal, RP. 1997. Linear Integral Equations, Birkhauser, Boston. 181-218.

Muhammad, US., Tamour, Zu. and Irfan, RA. 2013. A New Algorithm for Linear and Nonlinear Abel's Integral Equations. Int. J. of Modern Mathematical Sciences. 8(2): 123-129.

Muskhelishvilli, NI. 1963. Singular integral equations, Noordholf, Groningen. 113-162.

Olmstead , WE. and Handelsman, RA. 1976. Asymptotic solution to a class of nonlinear Volterra integral equations. II, SIAM J. Appl. Math. 30(1):180-189.

Polyanin, AD., Zaitsev, VF. 1999. Handbook of Exact Solutions for Ordinary Differential Equations. CRC Press, NewYork, USA. 29-37 .

Praveen, KG. 2012. Modified New Iterative Method for Solving Nonlinear Abel Type Integral Equations.International Journal of Nonlinear Science. 14:307-315.

Ripley, BD. 1981. Spatial Statistics, Wiley, New York, USA. 128-145.

Singh, OP., Singh, VK. and Pandey, RK. 2010. A stable numerical inversion of Abel integral equation using almost Bernstein operational matrix. Journal of Quantitative Spectroscopy & Radiative Transfer. 111:245-252.

Stoyan, D., Kendall, WS. and Mecke, J. 1987. Stochastic Geometry and its Applications. Akademie-Verlag, Berlin. 145-187.

Sunil, K., Singh, OP. and Sandeep, DI. 2011. Generalized Abel Inversion Using Homotopy Perturbation Method. Appl. Math. 2:254-257.

Teresa, DI., Pedro, Li. and Magda, RE. 2005. Computational Methods for a Nonlinear Volterra Integral Equation. Proceeding of Hercma. 100-107.

Vardi, Y., Shepp, LA. and Kaufman, L. 1985. A statistical model for positron emission tomography. J. Amer. Statist. Assoc. 80:8-37.

Wazwaz, AM. 2011. Linear and Nonlinear Integral Equations Method and applications. Beijing, Berlin. 592-594.

Wazwaz, AM. 1997. A First Course in Integral Equations, World Scientific, New Jersey, USA. 264-331.

Yousefi, SA. 2006. Numerical solution of Abel's integral equation by using Legendre wavelets. Appl. Math. Comput. 175:574-580.

Zeilon, N. 1924. Surquelques points de la theorie de l' equation integrale d'Abel. Arkiv. Mat. Astr. Fysik. 18:1-19.

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